MATHEMATICAL SIMULATION OF GRAVITATIONAL WAVES IN THE OCEAN IN THE APPROXIMATION OF "SHALLOW WATER"

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Problems of the behavior of gravitational waves in the approximation of "shallow water" – motion of a solitary wave over the water surface, exit of a wave to the shore, passage of a solitary wave over a submerged rock – are solved. The solution of the first problem showed that in modeling the motion of a solitary (soliton-type) wave the "shallow water" approximation breaks down at a ratio of water depth to wavelength equal to 0.3. An analysis of the results of solution of the second problem indicates that the "shallow water" approximation cannot be used for calculation of the height of a wave in its exit to the shore but it can be used for estimation of the distance from the shore where the wave is turned over. It follows from the solution of the third problem that the "shallow water" approximation is suited only for obtaining a qualitative picture of the distortion of the profile of a wave in its motion over a rock.

Several stages can be distinguished in a theoretical study of the tsunami phenomenon [1]:

1) description of the source of the disturbance and the motion of the water under the action of external forces (determination of the initial profile of the wave);

2) formation of a wave structure, which then moves as a whole;

3) propagation of the wave in a deep-water portion of the ocean;

4) incidence of the wave on a shallow-water near-shore portion of the ocean;

5) reflection of the wave from an uneven part of the bottom and artificial submerged structures.

In this paper attention was concentrated on items 2-5. It was assumed that the external action is caused by the fall of a celestial body into the water rather than by motion of the bottom (as is usually adopted) [2].

A tsunami can apparently arise as a result of the fall of asteroids and comets into water basins. Numerical simulation of the fall of an asteroid with a radius of 10 km, a substance density of 2.5 g/cm^3, and an energy of $6 \cdot 10^7$ Mton in TNT equivalent into an ocean with a depth of 5 km showed that about 12% of the kinetic energy of the body is transferred to a water layer, thus leading to the appearance of a powerful wave with a height of 25–35 km that is capable of disastrous consequences over the entire territory of the earth. In fact, the fall of such an asteroid to earth is unlikely. However, estimates of the consequences in the fall of an asteroid with a radius of 0.2 km showed that this event can be accompanied by catastrophic phenomena, because here a wave with an initial amplitude of about 0.3–0.8 km is formed [3].

The length of tsunami waves in the region of generation is usually close to the size of the site of the disturbance. Here it can amount to only several kilometers. The initial disturbances are grouped into a system consisting of two or three waves following each other in the open ocean. A difficulty in modeling this stage (item 2) is associated with the impossibility of using the "shallow water" approximation.

As the waves move, they become not too pronounced: their height (i.e., the vertical distance from the crest to the tough) amounts to several meters, and the length can reach tens or even hundreds of kilometers [4]. Even the deepest parts of the ocean turn out to be shallow for them, and modeling of this stage does not present special difficulties because the "shallow water" approximation can be used. The velocity of the wave is determined by the Lagrange formula: $v = \sqrt{gh}$. With a mean depth of the Pacific Ocean of about 4000 m, the theoretically calculated

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velocity of a tsunami wave is 716 km/h. This seems unlikely, but the maximum measured velocity of a tsunami wave was even higher – about 1000 km/h. In fact, the velocity of the majority of tsunami waves is somewhat below the theoretical value and ranges from 400 to 500 km/h.

On approaching the shore, a tsunami can grow from 1–2 m in the open ocean to several tens of meters on the shore depending on the coastal relief of the bottom and the shape of the shore line. But, the chief factor with which an increase in the height of the wave is associated is the decrease in the depth of the ocean. The latter can be calculated by the Airy–Green formula $h_{sh} = h_m \sqrt{H_m / H_{sh}}$ [4].

Having reached a shallow-water shelf, the wave becomes higher, heaves, and turns into a moving wall. In modeling this stage of wave motion, the "shallow water" approximation again becomes inapplicable.

Under certain conditions, a tsunami can evidently appear not in the form of a wave train but in the form of a solitary wave (soliton).

It is very difficult to predict the time of tsunami arrival (and the height of the waves) at certain parts of the shore. The point is that little is known about the manner in which the height of a tsunami wave in the first kilometers of its path and the velocity of wave propagation along different paths change depending on the relief of the ocean bottom. And it is extremely difficult to predict the behavior of a tsunami directly at shores having a complex shape, bays, and inlets.

The fact that tsunami are nonlinear waves also considerably hampers reliable prognoses [2].

In the present paper we used the traditional "shallow water" approximation to solve problems of the behavior of gravitational waves. However, as will be shown in what follows, the range of its applicability is rather limited in modeling complex wave phenomena in ocean waters.

Simplification of the System of Equations in the "Shallow Water" Approximation. The "shallow water" approximation means that the amplitude of the wave is much smaller than the depth of the basin ($\varepsilon = a/h << 1$) and the wavelength is much larger than the depth ($\delta = h/l << 1$) [5].

Since we are considering long waves, we can neglect the total acceleration along the OZ axis in the equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}, \nabla)\, \vec{u} + \frac{1}{\rho_0} \nabla P = - g\vec{j} \tag{1}$$

and can set

$$\frac{du_z}{dt} = \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_z \frac{\partial u_z}{\partial z} = 0 .$$

Then

$$\frac{1}{\rho_0} \frac{\partial P}{\partial z} = - g . \tag{2}$$

After integration over z with account for the boundary conditions we have

$$P = P_0 + \rho_0 g \,(z^s - z) . \tag{3}$$

We determine the derivative of the pressure entering the equation of motion

$$\frac{\partial P}{\partial x} = \frac{\partial P_0}{\partial x} + \rho_0 g \frac{\partial z^s}{\partial x} . \tag{4}$$

The right-hand side of (1) does not depend on the vertical coordinate in all changes of the liquid flow and the shape of the free surface. The pressure is eliminated from the equation of the dynamics using equality (4). Now the velocities are independent of the depth. Therefore, the derivative with respect to z disappears in the equation and, consequently, the vertical velocity no longer affects the horizontal flow. Thus, we have

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + g \frac{\partial z^s}{\partial x} = = -\frac{1}{\rho_0} \frac{\partial P}{\partial x}. \tag{5}$$

The second equation is obtained by integration of the continuity equation

$$\operatorname{div} \vec{u} = 0 \tag{6}$$

with account for the kinematic relations on the boundary surfaces, i.e., at the bottom and on the free surface. As a result, we obtain a new relation connecting the change in the height of the free surface to the nonuniformity of the horizontal flows under it:

$$\frac{\partial}{\partial x} \int_{-h}^{z^s} u_x dz + \frac{\partial z^s}{\partial t} = 0. \tag{7}$$

If

$$U = \int_{-h}^{z^s} u_x dz ,$$

then formula (7) determines the equilibrium between the flow rate through the surface of this column and the change in its own volume.

Since the horizontal velocity does not depend on the vertical coordinate, then (7) can be written in the form

$$\frac{\partial}{\partial x} u_x \int_{-h}^{z^s} dz + \frac{\partial z^s}{\partial t} = 0 \tag{8}$$

or simply

$$\frac{\partial}{\partial x} (u_x (h + z^s)) + \frac{\partial z^s}{\partial t} = 0. \tag{9}$$

Thus, in the "shallow water" approximation we obtain a one-dimensional system of equations:

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + g \frac{\partial z^s}{\partial x} = -\frac{1}{\rho_0} \frac{\partial P_0}{\partial x}, \quad \frac{\partial}{\partial x} (u_x (h + z^s)) + \frac{\partial z^s}{\partial t} = 0,$$

which can be used in modeling gravitational waves on water.

Motion of a Solitary Wave over a Water Surface. The motion of a solitary wave over the surface of water was calculated mainly to check the correctness of the operation of the computer program. Figure 1 presents results of calculations for a soliton on "shallow water." Wave profiles at different instants of time are shown. The initial profile has a height of 0.05 m, and the water depth is 1 m. The grid contained 100 nodes on the wave profile. It is seen from the figure that the calculated value virtually coincides with the analytical one.

Exit of a Solitary Wave to a Shore. Figure 2 presents the results of calculations of exit of a solitary wave to a shore that are obtained using the "shallow water" approximation. Wave profiles at different instants of time are shown. The initial profile has a height of 0.05 m, a water depth of 1 m, and a height of the wave before turnover of 0.071 m (a), a height of 5 m, a water depth of 100 m, a wavelength of about 1000 m, and a height of the wave before turnover of 7.18 m (b), and a height of 5 m, a water depth of 1000 m, a wavelength of about 4000 m, and a height of the wave before turnover of 10.01 m (c). The grid contained 100 nodes each on the wave profile and at the bottom.

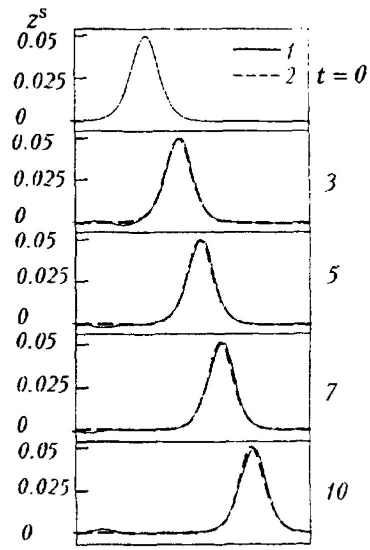

Fig. 1. Results of calculations for a soliton on "shallow water": 1) analytical solution; 2) calculation. z^s, m; t, sec.

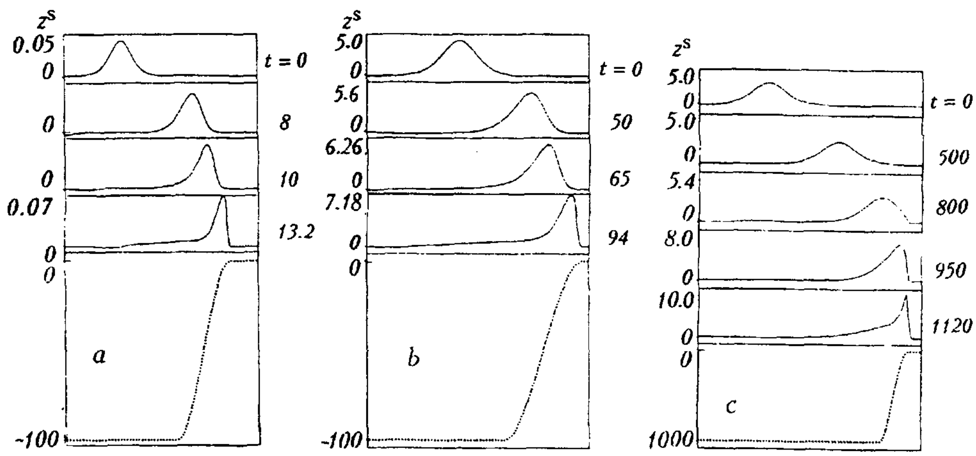

Fig. 2. Results of calculations for exit of a solitary wave to a shore (the dashed lines indicate the shape of the bottom).

It is seen from the figures presented that on approaching the shore the wave profile increases with a simultaneous increase in the steepness of the leading front, but at the shore the wave begins to turn over. The results obtained show that the longer the wave, the larger the height it has in exit to the shore.

Thus, the results of the solution of the problem of exit of a wave to the shore indicate that the "shallow water" approximation cannot be used for calculation of the wave height in its exit to the shore but it can be used for estimation of the distance to the shore where the wave turns over. Thus, the calculations showed that the height of the wave before turnover increases by only 40%, which does not correspond to natural observations.

Passage of a Solitary Wave over a Submerged Rock. Figure 3 presents results of calculations for passage of a solitary wave on the surface of a submerged obstacle that are obtained using the "shallow water" approximation. Wave profiles at different instants of time are shown. The initial profile has a height of 0.05 m, a water depth of 1 m, and a rock height of 0.5, 0.75, and 0.95, respectively. The maximum height of the wave is equal to 0.057 m (t = 7.2 sec) in Fig. 3a, 0.063 m (t = 7.4 sec) in Fig. 3b, and 0.072 m (t = 8.2 sec) in Fig. 3c. The grid contained 100 nodes each on the wave profile and at the bottom.

An analysis of the results indicates that the higher the rock, the higher the value attained by the height of the wave profile above the rock and the greater the role played by nonlinearity in the behavior of the wave. Thus, the results of solving the problem of passage of a solitary wave over a submerged rock showed that the "shallow water" approximation can be used only for obtaining a qualitative picture of the distortion of the profile of the wave in its motion over the rock.

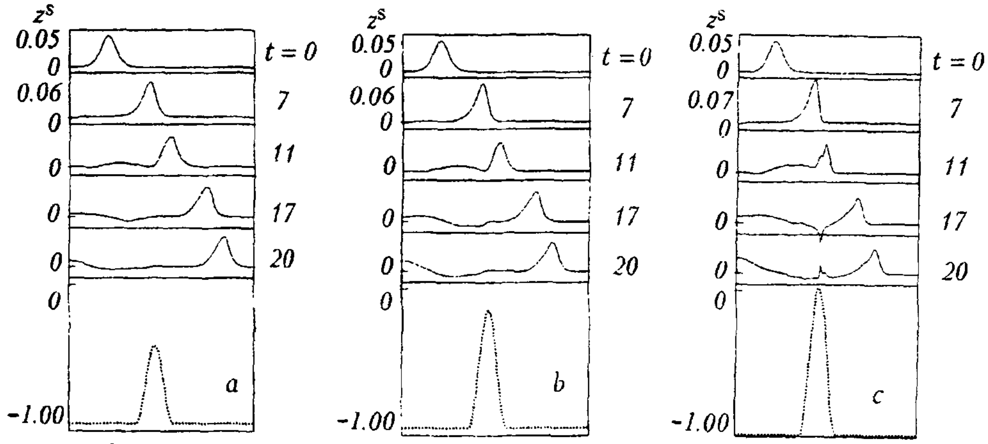

Fig. 3. Results of calculations for a solitary wave and a rock at the bottom.

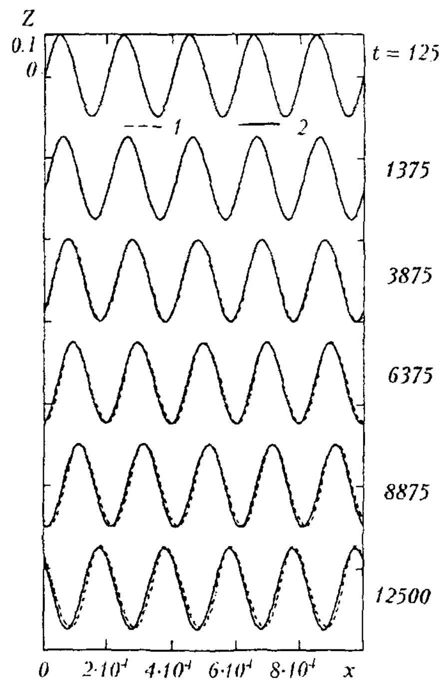

Fig. 4. Results of calculations for periodic gravitational waves on water for initial data corresponding to the linear case: 1) analytical solution; 2) calculation. Z, m.

Modeling of Periodic Gravitational Waves on Water. A zone of depth H is the region of the solution of this problem:

$$-\infty < x < \infty \, ; \quad -H < z < Z(x, t) \, .$$

Here $Z(x, t)$ is the unknown upper moving part of the boundary of the region. We seek it in the class of functions once continuously differentiable with respect to x and t that additionally possess the property of periodicity with respect to the variable x:

$$Z(x, t) = Z(x + L, t) \, .$$

Figure 4 presents results of calculations for initial data corresponding to a ratio of water depth to wavelength equal to $\delta = 0.01$ and a ratio of wave height to water depth equal to $\varepsilon = 0.01$, i.e., a virtually linear case. The initial profile has a height of 10 cm, a water depth of 100 m, and a wavelength of 10 km. Wave profiles are shown at different instants of time. The grid contained 151 nodes on the wave profile. It is seen from the results presented that the calculated value virtually coincides with the exact one.

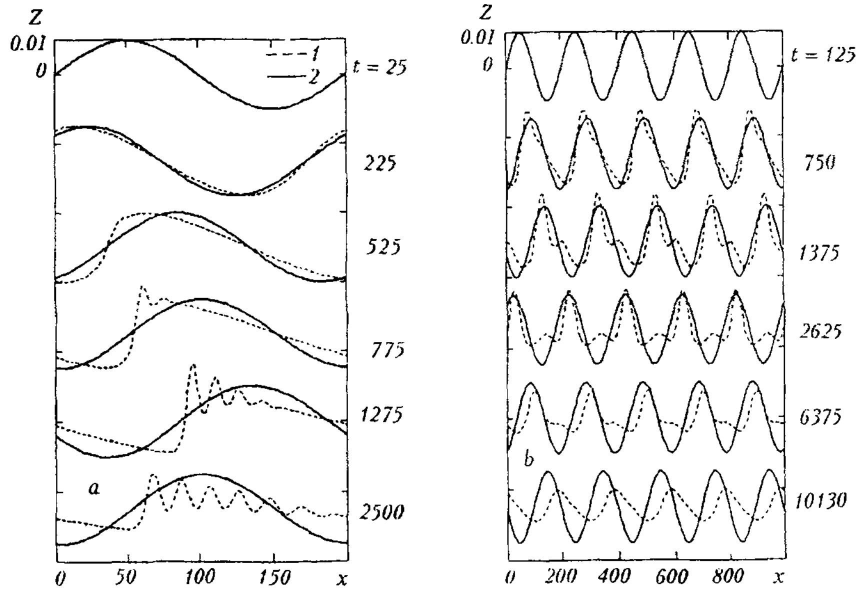

Fig. 5. Results of calculations for periodic gravitational waves on water for initial data corresponding to a case where nonlinearity begins to exert an effect: 1) analytical solution; 2) calculation.

Figure 5 presents results of calculations for a case where nonlinearity begins to exert an effect. The ratio of wave height to water depth is 0.01. In Fig. 5a (the initial profile has a height of 1 cm, a water depth of 1 m, and a wavelength of 100 m) wave profiles are shown at different instants of time. The grid contained 151 nodes on the wave profile. In Fig. 5b (the initial profile, water depth, and wavelength are the same) wave profiles are shown for other instants of time.

Thus, modeling of periodic gravitational waves on water showed that the "shallow water" approximation allows one to obtain a calculated solution that agrees well with the exact one for the linear case. However, the calculated solution in a case where nonlinearity begins to exert an effect does not reflect all the special features of the exact solution (the so-called Stokes wave).

NOTATION

v, velocity of wave motion in the Lagrange formula; g, free-fall acceleration; h, basin depth; H_{sh}, depth of the shallow-water region in the Airy–Green formula; h_{sh}, wave height in shallow water of depth H_{sh}; H_m, depth of the ocean in the Airy–Green formula; h_m, wave height at a depth of H_m; ε, δ, dimensionless small parameters; a, amplitude of the wave; l, wavelength; t, time variable; x, z, spatial variables; $\vec{u}$, vector of the liquid velocity; ρ_0, liquid density; P, pressure; $\vec{j}$, unit vector in the direction of the gravitational force; P_0, pressure of the surrounding atmosphere on the surface of the liquid; d/dt, substantial derivative; $\partial/\partial t$, $\partial/\partial z$, $\partial/\partial x$, partial derivatives with respect to the variables t, z, x, respectively; u_x, component of the velocity vector along the x axis; u_z, component of the velocity vector along the z axis; U, total flow rate of water through the surface of a unit column of liquid; H, depth of the basin in the problem of modeling periodic gravitational waves; L, period of the function $Z(x, t)$. Subscript: m, maximum. Superscript: s, boundary.

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